EXTENDED SPACE DUALITY IN THE NONCOMMUTATIVE PLANE

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Abstract:
Non-Commutative (NC) effects in planar quantum mechanics are investigated. We have constructed a *Master* model for a noncommutative harmonic oscillator by embedding it in an extended space, following the Batalin-Tyutin [5] prescription. Different gauge choices lead to distinct NC structures, such as NC coordinates, NC momenta or noncommutativity of a more general kind. In the present framework, all of these can be studied in a unified and systematic manner. Thus the dual nature of theories having different forms of noncommutativity is also revealed.

Keywords: Noncommutative quantum mechanics, Constraint systems, Batalin-Tyutin quantization.

Introduction: Non-Commutative (NC) Quantum Field Theories (QFT) [1] have created a lot of interest in the High Energy Physics community because of its direct connection to certain low energy limits of String theory [2]. Noncommutativity is induced in the open string boundaries that are attached to D-branes, in the presence of a two-form background field. This phenomenon in turn renders the D-branes into NC manifolds. Stringy effects are manifested in NCQFT framework. The major advantage of working in the latter is that the basic structure of QFT in conventional (commutative) spacetime remains intact and the fundamental (two-point) correlation functions of QFT are not modified by NC effects. This vital fact emerges from the construction of the NC generalization of a conventional QFT where the products of the field

operators in the QFT action are replaced by $*$ -product (or Moyal-Weyl product) defined below,

$$\hat{f}(x)*\hat{g}(x)=e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\sigma_\mu}\partial_{\xi_\nu}}\hat{f}(x+\sigma)\hat{g}(x+\xi)\mid_{\sigma=\xi=0}=\hat{f}(x)\hat{g}(x)+\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\hat{f}(x)\partial_\sigma\hat{g}(x)+\ O(\theta^2), \quad (1)$$

where $\theta_{\mu\nu}$ is conventionally taken as a constant anti-symmetric tensor. $\hat{f}(x)$ stands for the NC extension of $f(x)$. [1]

Exploiting the above rule (1), one derives the spacetime noncommutativity in $\hat{x}_\mu$-space,

$$[\hat{x}_\mu,\hat{x}_\nu]_*=\hat{x}_\mu*\hat{x}_\nu-\hat{x}_\nu*\hat{x}_\mu=i\theta_{\mu\nu}, \quad (2)$$

where $\hat{x}_\mu$ is the NC (operator) analogue of x_μ. In the present work we will only be concerned with spatial noncommutativity in a plane and so (2) reduces to

$$\{x^i,x^j\}=\theta\epsilon^{ij}\ ;\quad \epsilon^{12}=1. \quad (3)$$

Note that in our classical setup we will interpret the noncommutativity as a modification in the symplectic structure.

Effects of noncommutativity are often analyzed in terms of momentum variables [1]. This is because in the conventional forms of NC theory, the momentum degrees of freedom behave in a canonical way, obeying the symplectic structure,

$$\{\hat{p}_i,\hat{p}_j\}=0\ ;\quad \{\hat{x}_i,\hat{p}_j\}=\delta_{ij}. \quad (4)$$

Here the coordinates are noncommutative given by (3).

In case of NC quantum field theories, NC effects modify only the interaction vertices (through momentum dependent phase factors) when the fields are expressed by their Fourier transforms in terms of momentum variables.

On the other hand, in generic NC quantum mechanical models, one replaces [3, 4] the original coordinates by a canonical set of variables carrying a representation of the NC algebra. In terms of this new set of variables, the NC effects are manifestly present in the Hamiltonian. As a concrete example, the NC phase space algebra (3,4) can be simulated [3, 4] by the canonical variables (Q_i,P_j) obeying

$$\{Q_i,P_j\}=\delta_{ij},\ \{Q_i,Q_j\}=\{P_i,P_j\}=0,$$

with the identification,

$$x_i\equiv Q_i-\frac{\theta}{2}\epsilon_{ij}P_j\ ;\quad p_i\equiv P_i. \quad (5)$$

The NC extension of the ordinary oscillator Hamiltonian is,

$$\hat{H}=\frac{1}{2m}\hat{p}_i*\hat{p}_i+\frac{1}{2}\hat{x}_i*\hat{x}_i=\frac{1}{2m}\hat{p}_i\hat{p}_i+\frac{1}{2}\hat{x}_i\hat{x}_i,\quad i=1,2 \quad (6)$$

where we have used the following relations,

$$\hat{x}_i\hat{x}_i=\hat{x}_i\hat{x}_i+\frac{i}{2}\theta\epsilon_{ij}\delta_{ij}\equiv\ (\hat{x})^2;\quad \hat{p}_i*\hat{p}_i=\hat{p}_i\hat{p}_i\equiv(\hat{p})^2.$$

[1] $f(x)$ and $\hat{f}(x)$ are related in a non-trivial way by the Seiberg-Witten Map when quantum gauge theories are being considered. However, this is of no concern in the present work.

NC effects will become manifest in H once the $(\hat{x},\hat{p})$ variables are replaced by (Q,P) using (5).

Let us now define the perspective of our work. Essentially, we have generalized the above (P,Q)-model of NC harmonic oscillator to a *Master* model by embedding it in an extended phase space, in the Batalin-Tyutin (BT) framework [5]. The advantages of working with the *Master* model are the following: It can generate different structures of noncommutativity and ensures at the same time that the latter are gauge equivalent, (to be explained below). In contrast, note that the (Q,P)-system, as defined in (5) is geared to induce only the particular form of noncommutativity (3-4). However, keeping in mind subsequent (canonical) quantization of the model, we wish to maintain the canonical structure of the phase space of the new variables and BT formulation [5] is tailor made for that purpose.

BT prescription: Let us digress briefly on the BT formalism [5]. The basic idea is to embed the original system in an enlarged phase space (the BT space), consisting of the original "physical" degrees of freedom and auxiliary variables, in a particular way such that the resulting enlarged system possesses local gauge invariance. Imposition of gauge conditions accounts for the true number of degrees of freedom and at the same time the freedom of having different gauge choices leads to structurally distinct systems. However, all of them are assured to be gauge equivalent. In fact, this method is a generalization of the well known Stuckelberg formalism [6], which is applicable in the Lagrangian framework.

In the Hamiltonian analysis of constrained systems, as formulated by Dirac [7], the constraints are termed as First Class Constraints (FCC) if they commute (in the Poisson Bracket (PB) sense, modulo constraints) or Second Class Constraints (SCC) if they do not. The FCCs induce gauge invariance in the theory whereas the SCCs tend to modify the symplectic structure of the phase space for compatibility with the SCCs. The above modification induces a replacement of the PBs by Dirac Brackets (DB) as defined below,

$$\{A,B\}_{DB}=\{A,B\}-\{A,\psi_\alpha\}(\Psi_{\alpha\beta})^{-1}\{\psi_\beta,B\}. \tag{7}$$

where ψ_α refer to the SCCs and $\Psi_{\alpha\beta}\equiv\{\psi_\alpha,\psi_\beta\}$ is invertible on the constraint surface. The Dirac brackets are compatible with the SCCs so that the SCCs can be put "strongly" to zero.

Let us consider a generic set of constraints (ψ_α,ξ_l) and a Hamiltonian operator H with the following PB relations,

$$\{\psi_\alpha(q),\psi_\beta(q)\}\approx\Delta_{\alpha\beta}(q)\neq 0 \;\;;\;\; \{\psi_\alpha(q),\xi_l(q)\}\approx 0$$

$$\{\xi_l(q),\xi_n(q)\}\approx 0 \;\;;\;\; \{\xi_l(q),H(q)\}\approx 0. \tag{8}$$

In the above (q) collectively refers to the set of variables present prior to the BT extension and "$\approx$" means that the equality holds on the constraint surface. Clearly ψ_α and ξ_l are SCC and FCC respectively.

In systems with non-linear SCCs, (such as the present one), in general the DBs can become dynamical variable dependent due to the $\{A,\psi_\alpha\}$ and $\Delta_{\alpha\beta}$ terms, leading to problems for the quantization programme. This type of pathology is cured in the BT formalism in a systematic way, where one introduces the BT variables ϕ^α, obeying

$$\{\phi^\alpha,\phi^\beta\}=\omega^{\alpha\beta}=-\omega^{\beta\alpha}, \tag{9}$$

where $\omega^{\alpha\beta}$ is a constant (or at most a c-number function) matrix, with the aim of modifying the SCC $\psi_\alpha(q)$ to $\tilde{\psi}_\alpha(q,\phi^\alpha)$ such that,

$$\{\tilde{\psi}_\alpha(q,\phi),\tilde{\psi}_\beta(q,\phi)\}=0 \;\; ; \;\; \tilde{\psi}_\alpha(q,\phi)=\psi_\alpha(q)+\Sigma_{n=1}^{\infty}\tilde{\psi}_\alpha^{(n)}(q,\phi) \;\; ; \;\; \tilde{\psi}^{(n)}\approx O(\phi^n) \tag{10}$$

This means that $\tilde{\psi}_\alpha$ are now FCCs and in particular abelian. The explicit terms in the above expansion are,

$$\tilde{\psi}_\alpha^{(1)}=X_{\alpha\beta}\phi^\beta \;\; ; \;\; \Delta_{\alpha\beta}+X_{\alpha\gamma}\omega^{\gamma\delta}X_{\beta\delta}=0 \tag{11}$$

$$\tilde{\psi}_\alpha^{(n+1)}=-\frac{1}{n+2}\phi^\delta\omega_{\delta\gamma}X^{\gamma\beta}B_{\beta\alpha}^{(n)} \;\; ; \;\; n\geq 1 \tag{12}$$

$$B_{\beta\alpha}^{(1)}=\{\tilde{\psi}_\beta^{(0)},\tilde{\psi}_\alpha^{(1)}\}_{(q)}-\{\tilde{\psi}_\alpha^{(0)},\tilde{\psi}_\beta^{u(1)}\}_{(q)} \tag{13}$$

$$B_{\beta\alpha}^{(n)}=\Sigma_{m=0}^{n}\{\tilde{\psi}_\beta^{(n-m)},\tilde{\psi}_\alpha^{(m)}\}_{(q,p)}+\Sigma_{m=0}^{n}\{\tilde{\psi}_\beta^{(n-m)},\tilde{\psi}_\alpha^{(m+2)}\}_{(\phi)} \;\; ; \;\; n\geq 2 \tag{14}$$

In the above, we have defined,

$$X_{\alpha\beta}X^{\beta\gamma}=\omega_{\alpha\beta}\omega^{\beta\gamma}=\delta_\alpha^\gamma\delta. \tag{15}$$

A very useful idea is to introduce the improved function $\tilde{f}(q)$ corresponding to each $f(q)$,

$$\tilde{f}(q,\phi)\equiv f(\tilde{q})=f(q)+\Sigma_{n=1}^{\infty}\tilde{f}(q,\phi)^{(n)} \;\; ; \;\; \tilde{f}^{(1)}=-\phi^\beta\omega_{\beta\gamma}X^{\gamma\delta}\{\psi_\delta,f\}_{(q)} \tag{16}$$

$$\tilde{f}^{(n+1)}=-\frac{1}{n+1}\phi^\beta\omega_{\beta\gamma}X^{\gamma\delta}G(f)_\delta^{\lambda(n)} \;\; ; \;\; n\geq 1 \tag{17}$$

$$G(f)_\beta^{(n)}=\Sigma_{m=0}^{n}\{\tilde{\psi}_\beta^{(n-m)},\tilde{f}^{(m)}\}_{(q)}+\Sigma_{m=0}^{(n-2)}\{\tilde{\psi}_\beta^{(n-m)},\tilde{f}^{(m+2)}\}_{(\phi)}+\{\tilde{\psi}_\beta^{(n+1)},\tilde{f}^{(1)}\}_{(\phi)} \tag{18}$$

which have the property $\{\tilde{\psi}_\alpha(q,\phi),\tilde{f}(q,\phi)\}=0$. Thus, in the BT space, the improved functions are FC or equivalently gauge invariant. Note that $\tilde{q}$ corresponds to the improved variables for q. The subscript (ϕ) and (q) in the PBs indicate the variables with respect to which the PBs are to be taken. It can be proved that extensions of the original FCC ξ_l and Hamiltonian H are simply,

$$\tilde{\xi}_l=\xi(\tilde{q}) \;\; ; \;\; \tilde{H}=H(\tilde{q}). \tag{19}$$

One can also reexpress the converted SCCs as $\tilde{\psi}_\alpha^\mu\equiv\psi_\alpha^\mu(\tilde{q})$. The following identification theorem holds,

$$\{\tilde{A},\tilde{B}\}=\{A,\tilde{B}\}_{DB} \;\; ; \;\; \{\tilde{A},\tilde{B}\}\mid_{\phi=0}=\{A,B\}_{DB} \;\; ; \;\; \tilde{0}=0. \tag{20}$$

Hence the outcome of the BT extension is the closed system of FCCs with the FC Hamiltonian given below,

$$\{\tilde{\psi}_\alpha,\tilde{\psi}_\beta\}=\{\tilde{\psi}_\alpha,\tilde{\xi}_l\}=\{\tilde{\psi}_\alpha^\mu,\tilde{H}\}=0 \;\; ; \;\; \{\tilde{\xi}_l,\tilde{\xi}_n\}\approx 0 \; ; \; \{\tilde{\xi}_l,\tilde{H}\}\approx 0. \tag{21}$$

In general, due to the non-linearity in the SCCs, the extensions in the improved variables, (and subsequently in the FCCs and FC Hamiltonian), may turn out to be infinite series. This type of situation has been encountered before [8]. However, this is not the case in the present work.

NC harmonic oscillator: We will work with a specific model [9] for an NC harmonic oscillator,

$$L = q_i \dot{x}_i + \frac{\theta}{2}\epsilon_{ij} q_i \dot{q}_j - \frac{k}{2}x^2 - \frac{1}{2m}q^2. \tag{22}$$

The connection between L in (22) and the conventional models of noncommutativity arising in the coordinates of charged particle moving in a plane with a normal magnetic field is discussed in [9]. This first order Lagrangian possesses the following set of constraints,

$$\psi_1^i \equiv \pi_x^i - q^i \ ; \quad \psi_2^i \equiv \pi_q^i + \frac{\theta}{2}\epsilon^{ij} q^j \tag{23}$$

and the non-singular nature of the commutator matrix for the constraints

$$\Psi_{\alpha\beta}^{ij} = \{\psi_\alpha^i, \psi_\beta^j\} \ ; \quad \alpha, \beta \equiv 1, 2 \tag{24}$$

where

$$\Psi_{\alpha\beta}^{ij} = \begin{pmatrix} 0 & -\delta^{ij} \\ \delta^{ij} & \theta\epsilon^{ij} \end{pmatrix}$$

indicates that the constraints are Second Class constraints (SCC) in the Dirac terminology [7] The SCCs require a change in the symplectic structure in the form of Dirac Brackets [7] defined in (7). In the present case, the inverse of the constraint matrix (24)

$$\Psi_{\alpha\beta}^{(-1)ij} = \begin{pmatrix} \theta\epsilon^{ij} & \delta^{ij} \\ -\delta^{ij} & 0 \end{pmatrix}$$

leads to the following set of Dirac brackets,

$$\{x^i, x^j\}_{DB} = \theta\epsilon^{ij} \ , \quad \{x^i, q^j\}_{DB} = \delta^{ij} \ ; \quad \{q^i, q^j\}_{DB} = 0,$$
$$\{q^i, \pi_q^j\}_{DB} = 0 \ ; \quad \{x^i, \pi_x^j\}_{DB} = \delta^{ij} \ ; \quad \{\pi_x^i, \pi_q^j\}_{DB} = 0. \tag{25}$$

The Hamiltonian is

$$H = \frac{k}{2}x^2 + \frac{1}{2m}q^2. \tag{26}$$

Note the spatial noncommutativity and also the fact that q^i behaves effectively as the conjugate momentum to x^i. Although, explicit θ-dependence does not show up in the Hamiltonian, it will appear in the Hamiltonian equations of motion where the Dirac brackets are to be used. The model does not have any FCC in the physical space.

BT extension of NC harmonic oscillator: Let us now turn to the main body of our work. It is important to remember that the extended space is *completely* canonical and the Dirac brackets of (25) are not to be used. This is because the SCCs (23) are absent in the extended space and their place is taken by the FCCs that we derive below.

Following the procedure outlined in the previous section, we introduce a canonical set of auxiliary variables

$$\{\phi_\alpha^i, \phi_\beta^j\} = \epsilon_{\alpha\beta}\delta^{ij} \ , \alpha, \beta = 1, 2 \ ; \quad \epsilon_{12} = 1, \tag{27}$$

it is possible to convert the SCCs in (23) to the following FCCs $\tilde{\psi}^i_\alpha$,

$$\tilde{\psi}^i_1 \equiv \psi^i_1 + \phi^i_2 = \pi^i_x - q^i + \phi^i_2,$$
$$\tilde{\psi}^i_2 \equiv \psi^i_2 - \phi^i_1 - \frac{\theta}{2}\epsilon^{ij}\phi^j_2 = \pi^i_q + \frac{\theta}{2}\epsilon^{ij}q^j - \phi^i_1 - \frac{\theta}{2}\epsilon^{ij}\phi^j_2, \tag{28}$$

so the $\tilde{\psi}^i_\alpha$ are commutating,

$$\{\tilde{\psi}^i_\alpha, \tilde{\psi}^j_\beta\} = 0. \tag{29}$$

Thus the embedded model in extended space possesses local gauge invariance. Let us construct the improved variables $\tilde{q}$ as defined in (16-18). The connection between the operator algebra in any reduced (*i.e.* gauge fixed) offspring with its analogue in the gauge invariant parent model is the following identity:

$$\{A, B\}_{DB} = \{\tilde{A}, \tilde{B}\}. \tag{30}$$

In the present case we compute FC counterparts of x^i and q^i:

$$\tilde{q}^i = q^i - \phi^i_2 \ , \quad \tilde{x}^i = x^i - \phi^i_1 + \frac{\theta}{2}\epsilon^{ij}\phi^2_j,$$
$$\tilde{\pi}^i_x = \pi^i_x \ , \quad \tilde{\pi}^i_q = \pi^i_q - \phi^i_1, \tag{31}$$

which in turn generates the FC Hamiltonian

$$\tilde{H} = \frac{k}{2}\tilde{x}^2 + \frac{1}{2m}\tilde{q}^2 = \frac{k}{2}(x^i - \phi^i_1 + \frac{\theta}{2}\epsilon^{ij}\phi^j_2)^2 + \frac{1}{2m}(q^i - \phi^i_2)^2. \tag{32}$$

The FCCs take a simpler form,

$$\tilde{\psi}^i_1 = \tilde{\pi}^i_x - \tilde{q}^i \ ; \quad \tilde{\psi}^i_2 = \tilde{\pi}^i_q + \frac{\theta}{2}\epsilon^{ij}\tilde{q}^j. \tag{33}$$

This is the cherished form of the Hamiltonian where the NC correction has appeared explicitly in a fully canonical space with commuting space coordinates. However we would like to keep the set of dynamical equations [9]

$$\dot{q}^i = -kx^i \ ; \quad \dot{x}^i = \frac{1}{m}q^i + \theta k\epsilon^{ij}x^j \tag{34}$$

unchanged and this can be achieved by adding suitable terms in the Hamiltonian that are proportional to the FCCs. We construct $\tilde{H}_{Total}$

$$\tilde{H}_{Total} = \tilde{H} + \lambda^i_1\tilde{\psi}^i_1 + \lambda^i_2\tilde{\psi}^i_2 \tag{35}$$

and identify the arbitrary multipliers λ^i_α from (35):

$$\dot{q}^i = \{q^i, \tilde{H}_{Total}\} = \lambda^i_2 \ ; \quad \dot{x}^i = \{x^i, \tilde{H}_{Total}\} = \lambda^i_1. \tag{36}$$

Hence the final form of the gauge invariant FC Hamiltonian generating the correct dynamics of [9] is

$$\tilde{H}_{Total} = \frac{k}{2}(x^i - \phi^i_1 + \frac{\theta}{2}\epsilon^{ij}\phi^j_2)^2 + \frac{1}{2m}(q^i - \phi^i_2)^2$$

$$+(\frac{1}{m}q^i+\theta k\epsilon^{ij}x^j)(\pi_x^i-q^i+\phi_2^i)-kx^i(\pi_q^i+\frac{\theta}{2}\epsilon^{ij}q^j-\phi_1^i-\frac{\theta}{2}\epsilon^{ij}\phi_2^j). \quad (37)$$

This is the *Master* model that we advertised in the Introduction and constitute the main result of the present paper. The local gauge invariance in the enlarged space allows us to choose gauge conditions (according to our convenience) which in turn gives rise to different forms of symplectic structures and Hamiltonians, that , however, are gauge *equivalent*. This is the duality between different structures of noncommutativity, referred to earlier. As an obvious gauge choice, the unitary gauge

$$\psi_3^i\equiv\phi_1^i\ ,\quad \psi_4^i\equiv\phi_2^i \quad (38)$$

restricts the system to the original physical subspace with the spatial noncommutativity as given in (25) being induced by the full set of SCCs (the FCCs $\tilde{\psi}_1^i,\tilde{\psi}_2^i$ and the gauge fixing constraints ψ_3^i,ψ_4^i).

On the other hand, the following non-trivial gauge

$$\psi_3^i\equiv\phi_2^i\ ;\quad \psi_4^i\equiv\phi_1^i-\frac{\theta}{2}\epsilon^{ij}\phi_2^j+c\epsilon^{ij}\pi_x^j, \quad (39)$$

with c being an arbitrary parameter, generates the constraint matrix

$$\Psi_{\alpha\beta}^{ij}=\{\tilde{\psi}_\alpha^i,\psi_\beta^j\} \quad (40)$$

where,

$$\Psi_{\alpha\beta}^{ij}=\begin{pmatrix}0&0&0&-\delta^{ij}\\0&0&-\delta^{ij}&0\\0&\delta^{ij}&0&-\delta^{ij}\\\delta^{ij}&0&\delta^{ij}&\theta\epsilon^{ij}\end{pmatrix}.$$

The inverse matrix

$$\Psi_{\alpha\beta}^{(-1)ij}=\begin{pmatrix}\theta\epsilon^{ij}&\delta^{ij}&0&\delta^{ij}\\-\delta^{ij}&0&\delta^{ij}&0\\0&-\delta^{ij}&0&0\\-\delta^{ij}&0&0&0\end{pmatrix}$$

induces the Dirac brackets,

$$\{x^i,x^j\}_{DB}=(\theta+2c)\epsilon^{ij}\ ;\quad \{x^i,\pi_x^j\}_{DB}=\delta^{ij}\ ;\quad \{\pi_x^i,\pi_x^j\}_{DB}=0. \quad (41)$$

The choice $c=-\theta/2$ reduces the phase space to a canonical one with the Hamiltonian

$$H=\frac{k}{2}x^2+(\frac{1}{2m}+\frac{k\theta^2}{8})\pi_x^2-\frac{k\theta}{2}\epsilon^{ij}x^i\pi_x^j. \quad (42)$$

Rest of the non-trivial Dirac brackets are not important in the present case. This structure is in fact identical to the one studied in [4].

Another interesting gauge is

$$\psi_3^i\equiv q^i-\phi_2^i+ax^i\ ;\quad \psi_4^i\equiv x^i-\phi_1^i+\frac{\theta}{2}\epsilon^{ij}\phi_2^j+bq^i, \quad (43)$$

where a and b are arbitrary parameters. Once again, the subsequent constraint matrix

$$\Psi^{ij}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & -a\delta^{ij} & 0 \\ 0 & 0 & 0 & -b\delta^{ij} \\ a\delta^{ij} & 0 & 0 & -\delta^{ij} \\ 0 & b\delta^{ij} & \delta^{ij} & \theta\epsilon^{ij} \end{pmatrix}$$

and its inverse

$$\Psi^{(-1)ij}_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{\delta^{ij}}{ab} & \frac{\delta^{ij}}{a} & 0 \\ \frac{\delta^{ij}}{ab} & \frac{\theta\epsilon^{ij}}{b^2} & 0 & \frac{\delta^{ij}}{b} \\ -\frac{\delta^{ij}}{a} & 0 & 0 & 0 \\ 0 & -\frac{\delta^{ij}}{b} & 0 & 0 \end{pmatrix}$$

results in the symplectic structure,

$$\{x^i, x^j\}_{DB} = 0 \ ; \quad \{q^i, q^j\}_{DB} = \frac{\theta}{b^2}\epsilon^{ij} \ ; \quad \{x^i, q^j\}_{DB} = -\frac{\delta^{ij}}{ab}. \tag{44}$$

The choice of the parameters $a = \pm 1, b = \mp 1$ fixes q^i to be the conjugate momentum to x^i but *the momentum variables have now become noncommutative*. Comparing the Hamiltonian

$$H = \frac{k}{2}q^2 + \frac{1}{2m}x^2 \tag{45}$$

with (26) (where x^i were noncommutative) one observes that q^i and p^i have replaced one another. This exercise clearly demonstrates the fact that coordinate or momentum noncommutativity are actually different sides of the same coin and are connected by gauge transformations.

We conclude with a brief comment on the angular momentum L of the system. Remembering that in the extended space, the symplectic structure is completely canonical, L will have the obvious form

$$L = \epsilon^{ij}(x^i\pi^j_x + q^i\pi^j_q + \phi^i_1\phi^j_2). \tag{46}$$

Upon utilizing the canonical commutators, L will generate correct transformations on the degrees of freedom. But notice that L is not gauge invariant (*i.e.* FC) in the extended space. Its FC generalization will be,

$$L = \epsilon^{ij}(\tilde{x}^i\tilde{\pi}^j_x + \tilde{q}^i\tilde{\pi}^j_q), \tag{47}$$

since $\tilde{\phi}^i_\alpha = 0$. To recover the correct transformations of the physical variables one has to construct L_{Total},

$$L_{Total} = L + \xi^i_\alpha\tilde{\psi}^i_\alpha,$$

$$\xi^i_1 = \epsilon^{ij}(-\phi^j_1 + \frac{\theta}{2}\epsilon^{jk}\phi^k_2) \ ; \quad \xi^i_2 = -\epsilon^{ij}\phi^j_2. \tag{48}$$

In the unitary gauge $\tilde{L}_{Total}$ reduces to the expression given in [9]. For other gauge choices the structure of angular momentum will be different and the corresponding Dirac brackets will reproduce the transformation.

Conclusion: We have generalized the noncommutative harmonic oscillaor to a *Master* model having gauge invariance, in a Batalin-Tyutin extended space Hamiltonian framework.

The NC effects manifest themselves through the embedding procedure. We have shown that different gauge choices lead to different Hamiltonian systems with distinct symplectic structures. This clearly demonstrates the duality (or gauge equivalence) between different types of noncommutativity, (such as between coordinates, between momenta and between both coordinates as well as momenta), that are induced by different gauge choices in the same *Master* model.

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